

The Proof of the Non-periodicity of Four Kinds of Compound Trigonometric Functions

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Abstract—The non-periodicity of the function is an important part of the field of mathematics. First of all, this paper sums up a method to prove that the $\sin x^\alpha$ is a non-periodic function starting from the periodic definition, through the thought of the classified discussion, using the knowledge that the rational number is not equal to the irrational number in the real number, and the existing theorem conclusions. Secondly, another method of proving the non-periodic of $\sin x^\alpha$ function is obtained, based on ideas of number form combining, through the theorem that the limiting value of the adjacent intersections of graphics with the x axis is 0. Finally, with ideas of analogical proof method, the conclusion that $\cos x^\alpha$, $\tan x^\alpha$ and $\cot x^\alpha$ are non-periodic functions are proved.

Keywords—Non-periodic Function, Classification and Discussion, Combination of Number and Shape, Function Limit, Compound Trigonometric Function.

I. INTRODUCTION

The periodicity of the function is an important nature in the elementary function of our middle school. It has a close connection with the symmetry and monotonicity of the function. The study of the periodic problem [1-4] involves many aspects. In recent years, the study of the non-periodicity of the function also has some important achievements.

In 1982, Tang [5] adopted two methods to prove that $\sin x^2$ is not a periodic function: the first method is the proof method by contradiction, and the second method uses the non-countable knowledge of the real number set. In 1988, Chen [6] proved that $\sin x^\alpha$ is not a periodic function. In 1995, Tu [7] proved that both compound cosine function $\cos(\Phi(x))$ and compound tangent function $\tan(\Phi(x))$ are not periodic functions. In 2008, Li and Liu [8] gave the condition under which $\sin \Phi(x)$ is not a periodic function. Furthermore, the periodic problems of complex functions are discussed in [9-13].

II. PREPARATORY KNOWLEDGE

Lemma 1 [9] Functions with finite discontinuity points are aperiodic functions.

Lemma 2 [9] Let $f(x) \neq a$, a be a constant, and

$$\lim_{x \rightarrow \infty} f(x) = a.$$

Then $f(x)$ is not a periodic function.

Lemma 3 [9] Suppose that the distances between adjacent intersection points of the graph of $f(x)$ and the x axis are d_n , and $\lim_{n \rightarrow \infty} d_n = 0$ or $\lim_{n \rightarrow \infty} d_n = \infty$, then $f(x)$ is not a periodic function.

Lemma 4 [8] Suppose that the function $f(x) = \sin \Phi(x)$ is defined on the set A. If

$\Phi'(x)$ is continuous function on the set A;

$|\Phi'(x)|$ is not a periodic function on the set A;

Then $f(x) = \sin \Phi(x)$ is not a periodic function.

Lemma 5 [7] Functions that are not defined at finite points must be aperiodic functions.

Lemma 6 [6] If a function $f(x)$ is not differentiable only at finite points, then the function $f(x)$ is not a periodic function..

Lemma 7 [6] If the derived function $f'(x)$ of a function $f(x)$ is continuous, and $f'(x)$ is not a periodic function, then $f(x)$ is not a periodic function.

Lemma 8 [13] Suppose that the function $f(x) = \cos \Phi(x)$ is defined on the set A. If

$\Phi'(x)$ is continuous function on the set A;

$|\Phi'(x)|$ is not a periodic function on the set A;

Then $f(x) = \cos \Phi(x)$ is not a periodic function.

Lemma 9 [7] Suppose that the function $f(x) = \tan \Phi(x)$ is defined on the set A. If

$\Phi(x)$ is differentiable on the set A;

$|\Phi'(x)|$ is not a periodic function on the set A;

Then $f(x) = \tan \Phi(x)$ is not a periodic function.

Lemma 10 [7] Suppose that the function $f(x) = \cot \Phi(x)$ is defined on the set A. If

$\Phi(x)$ is differentiable on the set A;

$|\Phi'(x)|$ is not a periodic function on the set A;

Then $f(x) = \cot \Phi(x)$ is not a periodic function.

Lemma 11 [9] Suppose that the function $f(x)$ is bounded on any bounded interval, and there are point column $\{x_n\}$ such that $\lim_{n \rightarrow \infty} f(x_n) = \infty$, then $f(x)$ is not a periodic function.

III. MAIN RESULTS AND PROOF

Theorem 1 Let $f(u) = \sin u$ and $g(x) = x^\alpha (\alpha \neq 1)$, then $f(g(x)) = \sin x^\alpha$ is not a periodic function.

Proof method 1. When $\alpha > 0$ and $\alpha \neq 1$. If $(g(x)) = \sin x^\alpha$ is a periodic function. Then there exists $T \neq 0$, such that

$$\sin(x + T)^\alpha = \sin x^\alpha. \quad (1)$$

Let $x = 0$ in Eq 1, we have $\sin T^\alpha = 0$. i.e.

$$T^\alpha = k\pi \quad (k \in \mathbb{Z}). \quad (2)$$

Let $x = \sqrt[\alpha]{2}T$ in Eq 1, we have

$$\sin(\sqrt[\alpha]{2} + 1)^\alpha T^\alpha = \sin(2T^\alpha). \quad (3)$$

Substituting Eq2 in Eq 3, we have

$$\sin \left[(\sqrt[\alpha]{2} + 1)^\alpha k\pi \right] = \sin 2k\pi = 0. \quad (4)$$

$$(\sqrt[\alpha]{2} + 1)^\alpha k\pi = l\pi \quad (l \in \mathbb{Z}), \quad (5)$$

i.e.

$$(\sqrt[\alpha]{2} + 1)^\alpha = \frac{l}{k}. \quad (6)$$

No matter α is arbitrary real number, $(\sqrt[\alpha]{2} + 1)^\alpha$ is an irrational number. Eq 3 false. So $\sin x^\alpha$ is not a periodic function.

When $\alpha = 0$. Since the function $f(x) = \sin x^0$ has only one undefined point $x = 0$, $\sin x^\alpha$ is not a periodic function by Lemma 1.

When $\alpha < 0$. We have

$$f'(g(x)) = \alpha x^{\alpha-1} \cos x^\alpha = \frac{\alpha}{x^{1-\alpha}} \cos x^\alpha, x > 0$$

and

$$\lim_{x \rightarrow \infty} f'(g(x)) = 0$$

So $\sin x^\alpha$ is not a periodic function by Lemma 2.

Proof method 2. The intersection points of the graph of $\sin x^\alpha$ and the x axis are $(\sqrt[\alpha]{k\pi}, 0)$. Let the distances between adjacent intersection points be

$$d_n = \sqrt[\alpha]{(n+1)\pi} - \sqrt[\alpha]{n\pi}.$$

Since

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} (\sqrt[\alpha]{(n+1)\pi} - \sqrt[\alpha]{n\pi}) = 0,$$

$\sin x^\alpha$ is not a periodic function by Lemma 3.

Proof method 3. When $\alpha > 1$, the domain $A=\mathbb{R}$. Since $(x^\alpha)' = \alpha x^{\alpha-1}$, $\alpha x^{\alpha-1}$ is continuous function, and $|\alpha x^{\alpha-1}|$ is not a periodic function, $\sin x^\alpha$ is not a periodic function by Lemma 4.

When $\alpha < 1, \alpha \neq 0$, $(x^\alpha)' = \alpha x^{\alpha-1}$ is continuous on $(-\infty, 0) \cup (0, +\infty)$, is not defined at $x = 0$. $|(x^\alpha)'|$ is not a periodic function by Lemma 5. So $\sin x^\alpha$ is not a periodic function by Lemma 4.

When $\alpha = 0$. Since the function $f(x)=\sin x^0$ has an undefined point $x = 0$, $\sin x^\alpha$ is not a periodic function by Lemma 1.

Theorem 2 Let $f(u) = \cos u$ and $g(x) = x^\alpha (\alpha \neq 1, \alpha \neq 0)$, then $f(g(x)) = \cos x^\alpha$ is not a periodic function.

Proof method 1. When $\alpha > 1$. $f'(g(x)) = -\alpha x^{\alpha-1} \sin x^\alpha$. Since there are point column

$$x_n = (2n\pi + \pi/2)^{\frac{1}{\alpha}}, \quad (n = 1, 2, 3 \dots),$$

such that

$$\lim_{n \rightarrow \infty} f'(g(x_n)) = \lim_{n \rightarrow \infty} -\alpha(2n\pi + \pi/2)^{\frac{\alpha-1}{\alpha}} = -\infty.$$

So $f'(g(x))$ is not a periodic function by Lemma 10. Since $f'(g(x))$ is continuous, $f(g(x)) = \cos x^\alpha$ is not a periodic function by Lemma 7.

When $0 < \alpha < 1$. Since $\cos x^\alpha$ is not differentiable only at $x = 0$, $\cos x^\alpha$ is not a periodic function by Lemma 6.

When $\alpha < 0$. $\lim_{x \rightarrow \infty} f'(g(x))=0$, so $f'(g(x))$ is not a periodic function by Lemma 2. Since $f'(g(x))$ is continuous on $(0, +\infty)$, $f(g(x)) = \cos x^\alpha$ is not a periodic function by Lemma 7.

Proof method 2. Since $(\cos x^\alpha)' = -\alpha x^{\alpha-1} \sin x^\alpha$ is continuous and is not a periodic function, $\cos x^\alpha$ is not a periodic function by Lemma 7.

Theorem 3 Let $f(u) = \tan u$ and $g(x) = x^\alpha (\alpha \neq 1, \alpha \neq 0)$, then $f(g(x)) = \tan x^\alpha$ is not a periodic function.

Proof. The domain of $\tan x^\alpha$ is $A = (-\infty, +\infty) \setminus \{\sqrt[\alpha]{\pi/2 + k\pi}, k = 0, 1, 2, \dots\}$. $|g'(x)| = \alpha x^{\alpha-1}$ is not a periodic function, $g(x)$ is differentiable on the set A. So $f(g(x)) = \tan x^\alpha$ is not a periodic function by Lemma 9.

Theorem 4 Let $f(u) = \cot u$ and $g(x) = x^\alpha (\alpha \neq 1, \alpha \neq 0)$, then $f(g(x)) = \cot x^\alpha$ is not a periodic function.

Proof. The intersection points of the graph of $y = \cot x^\alpha$ and the x axis are $(\sqrt[\alpha]{\pi/2 + k\pi}, 0)$. Let the distances between adjacent intersection points be

$$d_n = \sqrt[\alpha]{\pi/2 + (n+1)\pi} - \sqrt[\alpha]{\pi/2 + n\pi}.$$

Since

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} (\sqrt[\alpha]{\pi/2 + (n+1)\pi} - \sqrt[\alpha]{\pi/2 + n\pi}) = 0,$$

$\cot x^\alpha$ is not a periodic function by Lemma 3.

IV. SUMMARY

This paper proves that $\sin x^\alpha, \cos x^\alpha, \tan x^\alpha$ and $\cot x^\alpha$ are non-periodic functions using ideas of number and form combining, the theorem that the limiting value of the distance of adjacent intersections of graphics with the x axis is 0, and theorems about non-periodic functions.

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